On viscosity and distribution solutions to PDEs with Neumann boundary conditions

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Outline

- Viscosity solutions of nonlinear PDEs with Neumann boundary conditions
- On relations of viscosity and distribution solutions.

Background

Consider the Neumann problem on a domain D:

$$\begin{cases} \frac{\partial u}{\partial t} = Lu(t,x), \quad t \geqslant 0, \quad x \in D, \\ u(T,x) = h(x), \quad x \in \bar{D}, \\ \frac{\partial u}{\partial \mathbf{n}} = 0, \quad x \in \partial D, \end{cases}$$

where $Lu = \frac{1}{2}\mathrm{tr}\sigma\sigma^*(t,x)\frac{\partial^2}{\partial x_i \partial x_j}u + b(t,x)\frac{\partial u}{\partial x_i}$.

When D is smooth and bounded, it is well known that under appropriate conditions the classical solution u can be represented by

$$u(t,x) = E\left[h(X^{t,x}(T))\right],$$

where $X^{t,x}$ is the reflected diffusion process starting from x at time t.

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where $X^{t,x}$ is the reflected diffusion process starting from x at time t.

What if one replaces Lu with a nonlinear term?

Background: Viscosity vanishing method

Consider

$$F(x, u, Du, D^2 u) = 0, \quad in\ U \subset \mathbb{R}^d. \tag{1}$$

F is continuous and degenerate elliptic, i.e.,

$$F(x, r, p, X) \leqslant F(x, r, p, Y), \quad \text{for all } x \in U,\ r \in \mathbb{R},\ p \in \mathbb{R}^d,$$

provided that $X \geqslant Y$.

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provided that $X\geqslant Y$.

Add a viscous term:

$$-\varepsilon\Delta u+F(x,u,Du,D^2u)=0, \quad \text{in } U.$$

Actually, if $u_\varepsilon\in\mathcal{C}^2(U)\cap\mathcal{C}(\bar{U})$ is a classical solution of the above equation and $u_\varepsilon\to u\in\mathcal{C}(U)$ in any compact subset of U. Then for any $\phi\in\mathcal{C}^2(U)$, $F(x,u(x),D\phi(x),D^2\phi(x))\leqslant 0$, provided that $u-\phi$ attains its maximum at $x\in U$.

Background: Viscosity Solution (M.G. Crandall and P.L.Lions)

- u is called a viscosity subsolution (supersolution resp.) of (1) if for any $\phi \in C^2(U)$, and if $u - \phi$ attains its local maximum (minimum resp.) at $x \in U$, the following holds:

$$F(x, u(x), D\phi(x), D^2\phi(x)) \leqslant (\geqslant \text{resp.}) 0.$$

Background: Viscosity Solution applied to stochastic control problem

Controlled diffusion process:

$X_s = x + \int_t^s b(X_r, \alpha_r)dr + \int_t^s \sigma(X_r, \alpha_r)dW_r,$

$\alpha \in \mathcal{A}$, $\mathcal{A}$: the set of adapted processes valued in a compact metric space.

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$\alpha \in \mathcal{A}$, $\mathcal{A}$: the set of adapted processes valued in a compact metric space.

The valued function defined by

$$V(t,x) := \inf_{\alpha \in \mathcal{A}} E[\int_t^T g(s, X_s, \alpha_s)ds + h(X_T)],$$

is a viscosity solution to the following nonlinear PDE (HJB equation):

$$-\frac{\partial V}{\partial t}(t,x) + \sup_{\iota \in U}\{-L_{t,x,\iota}V(t,x)dt - g(t,x,\iota)dt\} = 0,$$

where $L_{t,x,\iota}u = \frac{1}{2}\mathrm{tr}\sigma\sigma^*(t,x,\iota)\frac{\partial^2}{\partial x_i \partial x_j}u + b(t,x,\iota)\frac{\partial u}{\partial x_i}$.

Background: Neumann problem on D I

- P.Hsu (1985), T.S.Zhang (1990) Probabilistic approach to the Neumann problem.
- Pardoux-Zhang(1998), semilinear PDEs with nonlinear boundary conditions (generalized BSDEs, smooth domains).
- Zălinescu (2011), HJB equations (convex domains)
- Bahlali-Maticiuc-Zălinescu (2013), semilinear PDEs with nonlinear Neumann boundary conditions (BSDEs, smooth domains).
- J. Ren and W. (2013), HJB integro-differential equations (convex domains)
- X.Yang, T.S. Zhang (2014), semilinear elliptic PDEs and BSDEs with singular coefficients.

Background: Neumann problem on D II

- S.Tang and J.Li (2015) Stochastic optimal control problem of rsdes in convex domains
- C.Wong, X. Yang, J. Zhang (2022) Neumann problems for PDEs with nonlinear divergence terms.
- C.Wang, S. Yang, T.S.Zhang (2021) Reflected BM with singular drift.
- X.Huang and F.Wang (2022), F.Wang (2023) Distribution dependent RSDEs and nonlinear Neumann problem
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Conditions on D

Set for $x \in \partial D$,

$$N_x := \cup_{r>0} N_{x,r}$$

$$N_{x,r} := \{\mathbf{n} \in \mathbb{R}^d : |\mathbf{n}| = 1, \text{ and } B(x - r\mathbf{n}, r) \cap D = \emptyset\}.$$

(A) (Uniform exterior sphere condition) $\exists r_0 > 0$ s.t. $N_x = N_{x,4r_0} \neq \emptyset$, $\forall x \in \partial D$.

(B) $\exists \delta > 0$ and $\exists \beta \geqslant 1$ s.t. $\forall x \in \partial D$, there exists a unit vector l_x satisfying

$$\langle l_x, \mathbf{n} \rangle \geqslant \frac{1}{\beta}, \quad \forall \mathbf{n} \in \cup_{y \in B(x,\delta) \cap \partial D} N_y.$$

(C) $\exists f \in \mathcal{C}_b^2(\mathbb{R}^d)$ and a constant $\gamma > 0$ such that $\forall x \in \partial D$ and $\forall \mathbf{n} \in N_x$,

$$Df(x) \cdot \mathbf{n} \geqslant \frac{\gamma}{8r_0}.$$

Conditions of D

Remark

- (A) ensures that for any x satisfying $0 < d(x, \bar{D}) < 4r_0$, there exists a unique $\pi(x) \in \bar{D}$ s.t. $|x - \pi(x)| = d(x, \bar{D}), \quad \frac{\pi(x)-x}{|\pi(x)-x|} \in N_x$.
- (B) is satisfied if D is a Lipschitz domain.
- If D is convex, then (A) holds for any $r_0 > 0$ and (B) holds if $d = 2$ or D is bounded. (Tanaka, 1979)

Problem

We are concerned with the following HJB-type PDE with Neumann boundary:

$$\begin{cases} -\frac{\partial u}{\partial t} + H(t, x, Du, D^2 u) = 0 \quad \text{in} \quad (0, T) \times D, \\ -\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0, T) \times \partial D, \\ u(T, \cdot) = h(\cdot) \quad \text{on} \quad \bar{D}, \end{cases} \tag{2}$$

where

$$H(t, x, q, Q) := \max_{\iota \in U} \{ -\frac{1}{2} \mathrm{tr}(\sigma\sigma^*)(t, x, \iota) Q - \langle b(t, x, \iota), q \rangle - g(t, x, \iota) \},$$

$g, \ h \in \mathcal{C}_b$, U is a compact metric space.

Definition of viscosity solution

Set for $x \in \partial D$,

$$N_x^-(q) := \liminf_{\delta \downarrow 0} \{\langle q, -\mathbf{n}' \rangle : |x' - x| < \delta, x' \in \partial D \cup D_{2r_0}, \mathbf{n}' \in N_{\pi(x')}\},$$
$$N_x^+(q) := \limsup_{\delta \downarrow 0} \{\langle q, -\mathbf{n}' \rangle : |x' - x| < \delta, x' \in \partial D \cup D_{2r_0}, \mathbf{n}' \in N_{\pi(x')}\}.$$

(1) A function $u \in USC((0,T] \times \bar{D})$ is a viscosity subsolution of Eq.(2) if $u(T,x) \leqslant h(x)$ for $x \in \bar{D}$,

and for any $\phi \in \mathcal{C}^{1,2}((0,T) \times \bar{D})$, whenever $(t,x) \in (0,T) \times \bar{D}$ is a local maximum point of $u - \phi$, then

- if $(t,x) \in (0,T) \times D$, $-\frac{\partial \phi}{\partial t}(t,x) + H(t,x,D_x\phi, D_x^2\phi) \leqslant 0$,
- if $(t,x) \in (0,T) \times \partial D$,
  $\min\{-\frac{\partial \phi}{\partial t}(t,x) + H(t,x,D_x\phi,D_x^2\phi), N_x^-(D_x\phi)\} \leqslant 0$.

Definition of viscosity solution

(2) A function $u \in LSC((0, T] \times \bar{D})$ is a viscosity supersolution of Eq.(2) if $u(T, x) \geqslant h(x)$ for $x \in \bar{D}$,

and for any $\phi \in \mathcal{C}^{1,2}((0, T) \times \bar{D})$, whenever $(t, x) \in (0, T) \times \bar{D}$ is a local minimum of $u - \phi$, then

- if $(t, x) \in (0, T) \times D$, $-\frac{\partial \phi}{\partial t}(t, x) + H(t, x, u(t, x), D_x\phi, D_x^2\phi) \geqslant 0$,
- if $(t, x) \in (0, T) \times \partial D$,
  $\max\{-\frac{\partial \phi}{\partial t}(t, x) + H(t, x, u(t, x), D_x\phi, D_x^2\phi), N_x^+(D_x\phi)\} \geqslant 0$.

HJB equation with Neumann boundary

Theorem(Ren, W., 2019)

Assume

(H) σ and b are bounded, Lipschitz continuous and supported in $\overline{D} \cup D_{4r_0}$.

Then Eq.(2) has a unique viscosity solution u. Moreover, u has the following representation:

$$V(t,x) = \inf_{\alpha \in \mathcal{A}} E[\int_t^T g(s, X^{t,x}(s), \alpha(s))ds + h(X^{t,x}(T))], \tag{3}$$

where $X^{t,x}$ is the unique solution to the controlled reflected SDE:

$$X(s) = x + \int_t^s \sigma(X(r), \alpha(r))dW(r) + \int_t^s b(X(r), \alpha(r))dr + K(s),$$

$\alpha \in \mathcal{A}$: the set of progressively measurable processes taking values in a compact metric space U.

HJB equation with Neumann boundary

V is the corresponding value function satisfying that for every $(t, x) \in (0, T] \times \bar{D}$ and any stopping time τ valued in $[t, T]$,

(1) For all $\alpha \in \mathcal{A}$,

$$V(t,x) \leqslant E\Big(\int_t^\tau g(s, X^{t,x,\alpha}(s), \alpha(s))ds + V(\tau, X^{t,x,\alpha}(\tau)) \Big).$$

(2) For any $\delta > 0$, $\exists \alpha \in \mathcal{A}$ s.t.

$$V(t,x) + \delta \geqslant E\Big(\int_t^\tau g(s, X^{t,x,\alpha}(s), \alpha(s))ds + V(\tau, X^{t,x,\alpha}(\tau)) \Big).$$

HJB equation with Neumann boundary

Take $\varphi(x) := \rho(d(x,\bar{D})^2)$ where $\rho \in \mathcal{C}^1$, increasing, and

$$\rho(t) = \begin{cases} t, & t \in [0, 4r_0^2]; \\ 9r_0^2, & t \geqslant 9r_0^2. \end{cases}$$

Let $D_r := \{x \in \mathbb{R}^d, 0 < d(x,D) < r\}$.
(i) $\varphi \in C_b^1(\mathbb{R}^d)$, and $\nabla\varphi$ is bounded and Lipschitz continuous.
(ii) $\varphi(x) = |x - \pi(x)|^2$ for $x \in D_{2r_0}$.
Consider the penalized equation:

$$X_n^{t,x}(s) = x + \int_t^s b(X_n^{t,x}(r))dr + \int_t^s \sigma(X_n^{t,x}(r))dw(r) \quad (4)$$

$$-\frac{n}{2}\int_t^s \nabla\varphi(X_n(r))dr. \quad (5)$$

HJB equation with Neumann boundary

Proposition. (Ren, W. 2019)

Assume **(A)**-**(C)** and **(H)** hold. Then for any $T > 0$ and $R > 0$,

$$\sup_{(t,x,\alpha)\in[0,T]\times B(0,R)\times U} E[\sup_{t\leqslant s\leqslant T} |X_n^{t,x,\alpha}(s) - X^{t,x,\alpha}(s)|^2] \to 0.$$

Uniqueness of viscosity solution

Theorem.

Suppose u is a viscosity subsolution bounded from above, and v is a viscosity supersolution bounded from below of (2), then $u \leqslant v$ on $(0, T] \times \overline{D}$.

Test function:

$$\psi(t, s, x, y) := \frac{k}{2}(|x - y|^2 + |t - s|^2) - \varepsilon(f(x) + f(y)) + \delta_0(\frac{1}{2t} + \frac{1}{2s}).$$

Nonlinear PDEs with Neumann conditions

Consider the following nonlinear PDE with Neumann boundary condition in D:

$$\begin{cases} -\frac{\partial u}{\partial t}(t,x) - Lu(t,x)dt - f(t,x,u,\sigma^* Du(t,x)) = 0 \quad \text{in} \quad [0,T) \times D, \\ -\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad (0,T) \times \partial D, \\ u(T,x) = g(x) \quad x \in \bar{D}, \end{cases} \tag{6}$$

where D satisfies (A) and (B),

$$Lu = \frac{1}{2}\mathrm{tr}\sigma\sigma^*(x)\frac{\partial^2}{\partial x_i \partial x_j}u + b(x)\frac{\partial u}{\partial x_i}.$$

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where D satisfies (A) and (B),

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Aim: to present results of viscosity solution for (6).

Nonlinear PDEs with Neumann conditions

Let $\rho \in \mathcal{C}^1$ be a nondecreasing function s.t.

$$\rho(t) = \begin{cases} t, & t \in [0, 4r_0^2]; \\ 9r_0^2, & t \geqslant 9r_0^2. \end{cases}$$

and $\varphi(x) = \rho(d(x, \bar{D})^2)$.

Consider the following nonlinear PDE:

$$\begin{cases} -\frac{\partial u_n}{\partial t}(t,x) - Lu_n(t,x)dt - f(t,x,u_n,\sigma^* Du_n(t,x)) \\ \qquad\qquad + \frac{n}{2}\langle Du_n(t,x), \nabla\varphi(x)\rangle = 0, \quad \text{in} \quad [0,T) \times \mathbb{R}^d, \\ u_n(T,x) = g(x) \quad x \in \mathbb{R}^d, \end{cases} \tag{7}$$

Nonlinear PDEs with Neumann conditions

Assumptions.

(i) b, σ are bounded and continuous.

(ii) $\sigma\sigma^*(x) \geqslant a_0 I$ for some $a_0 > 0$.

(iii) $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$ are continuous and
$(y' - y)\big(f(t, x, y', z) - f(t, x, y, z)\big) \leqslant \alpha |y' - y|^2$ for some $\alpha \in \mathbb{R}$;
$|f(t, x, y, z) - f(t, x, y, z')| \leqslant l_1 |z - z'|$ for some $l_1 > 0$;
$|f(t, x, y, 0)| \leqslant l_2(1 + |y|), \quad |g(x)| \leqslant l_3(1 + |x|)$.

Theorem.(Wang, W., 2023)

Under Assumptions (i)-(iii), there exist functions $u_n : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ and $u : [0, T] \times \bar{D} \to \mathbb{R}$ s.t. u_n is a viscosity solution of (7), u is a viscosity solution of (6). Moreover,

$$u_n(t, x) \to u(t, x), \quad (t, x) \in [0, T] \times \bar{D}.$$

Some key results I

(1)、For every $n \geqslant 1$, the following equation has a weak solution $X_n^{t,x}$:

$$dX_n^{t,x}(s) = b(X_n^{t,x}(s))ds + \sigma(X_n^{t,x}(s))dW(s) - \frac{n}{2}\varphi(X_n^{t,x}(s))ds, \quad X_n^{t,x}(t) = x \in \bar{D}.$$

And $X_n^{t,x}, K_n^{t,x} := -\frac{n}{2}\int_t^{\cdot} \varphi(X_n^{t,x}(s))ds$ converge in distribution w.r.t. the uniform topology to $X^{t,x}$, $K^{t,x}$, which solves the following RSDE:

$$dX^{t,x}(s) = b(X^{t,x}(s))ds + \sigma(X^{t,x}(s))dW(s) + dK^{t,x}(s), \quad X^{t,x}(t) = x \in \bar{D}.$$

(2)、$(X^{t,x},\ K^{t,x})$ is cont. in distribution w.r.t. (t, x).

Some key results I

(3)、Denote by $(Y^{n,t,x}, Z^{n,t,x})$ and $(Y^{t,x}, Z^{t,x})$ the solutions to BSDEs:

$$Y_s^{n,t,x} = g(X^{n,t,x}(T)) + \int_s^T f(r, X^{n,t,x}(r), Y^{n,t,x}(r), Z^{n,t,x}(r))dr - \int_s^T Z^{n,t,x}(r)dW_r.$$

$$Y_s^{t,x} = g(X^{t,x}(T)) + \int_s^T f(r, X^{t,x}(r), Y^{t,x}(r), Z^{t,x}(r))dr - \int_s^T Z^{t,x}(r)dW_r.$$

Set

$$U^{n,t,x}(s) := \int_t^s Z^{n,t,x}(r)dW_r, \quad U^{t,x}(s) := \int_t^s Z^{t,x}(r)dW_r.$$

$(Y^{n,t,x}, U^{n,t,x})$ converges in distribution w.r.t. S-topology to $(Y^{t,x}, U^{t,x})$.

(4)、If $(t_n, x_n) \to (t, x)$, then there exists a subsequence $\{n_k\} \subset \{n\}$ s.t. $Y^{t_{n_k}, x_{n_k}}$ converges in distribution w.r.t. S-topology to $Y^{t,x}$.

The S-topology (Jakubowski, 1997) I

- $x_n, x_0 \in \mathcal{D}([0,1];\mathbb{R})$, x_n converges to x_0 in the S-topology if for each $\epsilon > 0$, $\exists v_{n,\epsilon} \in BV([0,1])$, $n = 0, 1, 2, \cdots$ s.t.

(1) $\sup_n \sup_{t\in[0,1]} |x_n(t) - v_{n,\epsilon}(t)| \leqslant \epsilon, \quad n = 0, 1, 2, \cdots$

(2) $v_{n,\epsilon} \xrightarrow{w*} v_{0,\epsilon}$.

The S-topology (Jakubowski, 1997) I

- For $\Gamma \subset \mathcal{D}([0,1];\mathbb{R})$, if for each $\epsilon > 0$ and each $x \in \Gamma$, $\exists v_{x,\epsilon} \in BV([0,1])$ and

$$\sup_{x\in\Gamma} \sup_{t\in[0,1]} |x(t) - v_{x,\epsilon}(t)| \leqslant \epsilon, \quad \sup_{x\in\Gamma} Var(v_{x,\epsilon}) < \infty. \tag{8}$$

Then there exists a sequence $\{x_n\} \subset \Gamma,\ x \in \mathcal{D}([0,1];\mathbb{R})$ such that $x_n \xrightarrow{S} x$.

- Conversely, if for every subsequence $\{x_n\} \subset \Gamma$, there exists a subsequence converging in the S-topology to some $x \in \mathcal{D}([0,1];\mathbb{R})$. Then (8) holds for the set Γ.

- On relations of viscosity and distribution solutions

Background

Consider the following PDE:

$$-\frac{1}{2}\mathrm{tr}\sigma\sigma^*(x)\frac{\partial^2}{\partial x_i \partial x_j}u - b(x)\frac{\partial u}{\partial x_i} + c(x)u = f(x), \quad \text{in} \quad D, \tag{9}$$

where D is an open, bounded domain and $b \in W^{1,\infty}(D)$, $\sigma \in W^{2,\infty}(D)$.

Definition of distribution solution

A function $u \in \mathcal{C}(\overline{D})$ is a distribution sub-solution (super-solution resp.) of (9) if for any $\phi \in \mathcal{D}_+(D) := \{\phi \in \mathcal{C}_0^2(D) | \phi \geqslant 0\}$,

$$\int_D (uL^*\phi - f\phi)\mathrm{d}x \leqslant (\geqslant \text{resp.}) 0$$

where L^* is the adjoint operator of L.

Background

- P.L.Lions (1984) proved that the following are equivalent:
  - (1) u is a viscosity subsolution of (9).
  - (2) u is a distribution subsolution of (9).
  - (3) $u \in C(D)$ and for all $\epsilon > 0$, any $d(x, \partial D) > \epsilon$, the following M is a submartingale:

$$M(t \wedge \tau_\epsilon) := u(X_{t\wedge\tau_\epsilon}) \exp\{-\int_0^{t\wedge\tau_\epsilon} c(X_r)dr\} + \int_0^{t\wedge\tau_\epsilon} f(X_s) \exp\{-\int_0^{s\wedge\tau_\epsilon} c(X_r)dr\}ds,$$

    where X is the diffusion process associated with L and τ_ϵ is the first exit time of X from $\{x \in D; d(x, \partial D) > \epsilon\}$.
- Ishii (1994) has presented an analytic proof for the equivalence between (1) and (2).

Problem

Consider the following Robin problem in a domain D:

$$\begin{cases} Lu(x) = -a_{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j} - b_i(x)\dfrac{\partial u}{\partial x_i} = f(x), & x \in D, \\ \mathfrak{B}u(x) = -a_{ij}(x)\dfrac{\partial u}{\partial x_j}\mathbf{n}_i(x) = 0, & x \in \partial D, \end{cases} \tag{10}$$

where D is of C^3, $\sigma^{ij} \in W^{2,\infty}(\bar{D})$, $b^i \in W^{1,\infty}(\bar{D})$, and $\mathbf{n}$: the inward unit normal vector at ∂D.

A function $u \in \mathcal{C}(\overline{D})$ is a distribution sub-solution (super-solution resp.) of (10) if

$$\int_D (uL^*\phi - f\phi)\mathrm{d}x \leqslant (\geqslant \text{resp.})0 \tag{11}$$

for any $\phi \in \mathcal{D}_+(D) := \{\phi \in \mathcal{C}_0^2(\overline{D}) | \mathcal{B}^*\phi = 0, \phi \geqslant 0\}$, where

$$L^*\phi = -\frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}\phi) + \frac{\partial}{\partial x_i}(b_i\phi), \quad \mathcal{B}^*\phi = -a_{ij}(x)\frac{\partial \phi}{\partial x_j}\mathbf{n}_i(x) - \langle \tilde{b}, \mathbf{n}\rangle\phi,$$

where $\tilde{b}_i := b_i - \frac{\partial a_{ij}}{\partial x_j}$.

Consider the SDE with oblique reflection in D:

$$\begin{cases} \mathrm{d}X_t = \sigma(X_t)\mathrm{d}W_t + b(X_t)\mathrm{d}t + K_t \qquad t \in [0, T] \\ X_0 = x \in \overline{D}, \\ K_t = \int_0^t \gamma(X_s)\mathrm{d}|K|_s, \qquad |K|_t = \int_0^t \mathbf{1}_{(X_s \in \partial D)}\mathrm{d}|K|_s, \end{cases}$$

where $\langle \gamma(x), \mathbf{n}(x) \rangle \geqslant \nu_0$ for some $\nu_0 > 0$.

Theorem (Ren-W.-Zheng, 2020)

If $u, f \in \mathcal{C}(\overline{D})$, the following are equivalent:

(1) u is a viscosity sub-solution (respectively, super-solution) of (10);

(2) For all $x \in \overline{D}$,

$$M_t := u(X_t) + \int_0^t f(X_s)\mathrm{d}s$$

is a sub-martingale (respectively, super-martingale).

- Comparison principle of viscosity solutions.
- Moment estimates of X.

$$E \sup_{s\in[0,t]} |X_s - x|^{2p} \leqslant C_p t^p, \quad E|K|_{s,t}^{2p} \leqslant C_p |t-s|^p.$$

Theorem (Ren-W.-Zheng, 2020)

Suppose $u, f \in \mathcal{C}(\overline{D})$ and $\langle \tilde{b}(x), \mathbf{n}(x) \rangle \leqslant 0$ for $x \in \partial D$. If M_t is a sub-martingale (respectively, super-martingale), then u is a distribution sub-solution (respectively, super-solution) of (10).

- Stability of solutions of RSDEs.
- Green's formula.
- Feymann Kac's representation.

Theorem (Ren-W.-Zheng, 2020) Suppose D is smooth sufficiently and $\langle \tilde{b}(x), \mathbf{n}(x) \rangle \leqslant 0$ for $x \in \partial D$, $u, f \in \mathcal{C}(\overline{D})$, and u is a distribution sub-solution (respectively, super-solution) of (10), then M_t is a sub-martingale (respectively, super-martingale).

Semilinear PDEs with Neumann boundary conditions

Now we consider the following semilinear PDE in an open, bounded and C^3 domain D:

$$\begin{cases} -\dfrac{\partial u}{\partial t} + Lu(t,x) = f(t,x,u(x)), \ \ (t,x) \in [0,T) \times D, \\ \mathfrak{B}u(t,x) = -a_{ij}(t,x)\dfrac{\partial u}{\partial x_j}\mathbf{n}_i(x) = 0, \ \ (t,x) \in [0,T) \times \partial D, \\ u(T,x) = g(x), \ \ x \in \bar{D}, \end{cases} \tag{12}$$

where D is of C^3, $Lu(t,x) := -a_{ij}(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} - b_i(t,x,u(t,x))\frac{\partial u}{\partial x_i}$.

A function $u \in \mathcal{C}([0, T] \times \overline{D})$ is called a distribution sub-solution of (12) if for any $\phi \in \mathcal{D}_+([0, T] \times D) := \{\phi \in \mathcal{C}^{1,2}([0, T) \times D) | \mathfrak{B}^* \phi = 0, \forall (t, x) \in [0, T) \times \partial D; \phi \geqslant 0\}$,

$$\int_D (u(T)\phi(T) - u(0)\phi(0))\mathrm{d}x + \int_0^T \int_D (-\frac{\partial \phi}{\partial t} u + uL^* \phi - f(t, x, u)\phi)\mathrm{d}x dt \leqslant 0 \tag{13}$$

where L^*, $\mathfrak{B}^*$ are the formal adjoint operators of L, $\mathfrak{B}$.

Assumptions

$\exists c > 0$, and $\alpha < 0$ satisfying that for any $t, s \in [0, T]$, $x,\ x' \in \mathbb{R}^d,\ y,\ y' \in \mathbb{R}$,

$$|b(t,x,y) - b(s,x',y')| + \|\sigma(t,x) - \sigma(s,x')\| \leqslant c(|t-s| + |x-x'| + |y-y'|);$$

$$\langle y - y', f(t,x,y) - f(t,x,y')\rangle \leqslant \alpha |y-y'|^2;$$

$$|f(t,x,y) - f(s,x',y)| \leqslant c(|t-s| + |x-x'|);$$

$$|g(x) - g(x')| \leqslant c|x-x'|.$$

It is known that when $\alpha \leqslant -\alpha_0$ for some α_0, the FBSDE admits a unique solution $(X^{t,x}, K^{t,x}, Y^{t,x}, Z^{t,x})$ **(Ma-Yong, 1999)**:

$$X_s = x + \int_t^s b(r, X_r, Y_r)dr + \int_t^s \sigma(r, X_r)dW_r + K_s, \quad s \in [t, T];$$

$$Y_s = g(X_T) + \int_s^T f(r, X_r, Y_r)dr - \int_s^T Z_r dW_r, \quad s \in [0, T].$$

And $u(t,x) := Y_t^{t,x}$ is a viscosity solution to Eq.(12).

Theorem(Ren, Wang, W., 2023)

Suppose the above assumptions hold. Then the following are equivalent:

(1) $u(t,x)$ is a viscosity subsolution to Eq.(12).

(2) $M_s := u(s, X_s^{t,x}) + \int_t^s f(r, X_r^{t,x}, u(r, X_r^{t,x}))dr$ is a submartingale.

If in addition, $\langle b(t,x,y) - \frac{\partial a_{ij}}{\partial x_j}, \mathbf{n}(x)\rangle \leqslant 0$ for $x \in \partial D$, then either of (1) or (2) implies that

(3) u is a distribution subsolution of Eq.(12).

- Comparison principle of viscosity solutions.
- Picard approximation results of FBSDEs.
- $(t,x) \to Y^{t,x}$ is continuous.

Thank you!